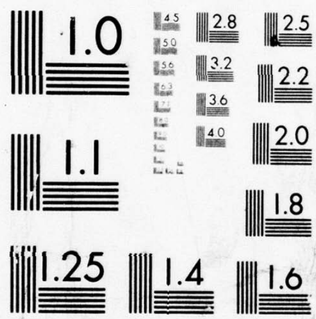
AD-A051 760 DEFENCE RESEARCH ESTABLISHMENT VALCARTIER (QUEBEC) F/G 20/3
SCATTERING OF ELECTROMAGNETIC BEAMS BY SPHERICAL OBJECTS,(U)
FEB 78 W G TAM, R CORRIVEAU

UNCLASSIFIED DREV-R-4086/78 NL

| OF |
AD
A051760

END
DATE
FILMED
4-78
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

UNCLASSIFIED

3

CRDV RAPPORT 4086/78
DOSSIER: 3633A-011
FÉVRIER 1978

DREV REPORT 4086/78
FILE: 3633A-011
FEBRUARY 1978

SCATTERING OF ELECTROMAGNETIC BEAMS BY SPHERICAL OBJECTS

W.G. Tam

R. Corriveau

Centre dé Recherches pour la Défense
Defence Research Establishment
Valcartier, Québec

BUREAU - RECHERCHE ET DEVELOPPEMENT
MINISTERE DE LA DÉFENSE NATIONALE
CANADA

RESEARCH AND DEVELOPMENT BRANCH
DEPARTMENT OF NATIONAL DEFENCE
CANADA

NON CLASSIFIÉ

CRDV R-4086/78 UNCLASSIFIED 14 DREV-R-4086/78
DOSSIER: 3633A-011 FILE: 3633A-011

6 SCATTERING OF ELECTROMAGNETIC BEAMS

BY SPHERICAL OBJECTS

by

10 W.G./Tam ~~and~~ R./Corriveau

12 29p.

CENTRE DE RECHERCHES POUR LA DEFENSE

DEFENCE RESEARCH ESTABLISHMENT

VALCARTIER

Tel: (418) 844-4271

Québec, Canada

11 Feb[redacted]78

404 945 EB

RESUME

La diffusion d'un faisceau gaussien TEM_{00} par un objet sphérique est traitée de façon exacte en termes de fonctions d'onde vectorielles sans aucune restriction de la dimension ou de la position du diffuseur. Les expressions obtenues pour la puissance absorbée et diffusée sont présentées sous forme de combinaisons linéaires des coefficients de Mie et peuvent être dès lors évaluées numériquement. Le problème correspondant à la diffusion d'un faisceau produit par un laser fonctionnant dans le mode TEM_{01}^{*} est aussi résolu. (NC)

ABSTRACT

sub ØØ

The scattering of a Gaussian beam TEM_{00} wave by a spherical object is treated exactly in terms of the vector wave functions without any restriction on the size or the position of the scatterer. Expressions obtained for the powers absorbed and scattered are given as linear combinations of the well-known Mie coefficients and can be readily applied to numerical computation. The corresponding problem for the scattering of a beam produced by a laser operating in the TEM_{01}^{*} mode is also solved. (U)

(TEM sub Ø1)*

TABLE OF CONTENTS

	RESUME/ABSTRACT	i
1.0	INTRODUCTION	1
2.0	EXPANSION OF THE ELECTRIC FIELD OF A GAUSSIAN TEM_{00} BEAM	2
3.0	SCATTERING OF A GAUSSIAN TEM_{00} BEAM	7
4.0	EXPANSION OF THE ELECTRIC FIELD OF A TEM_{01}^{*} BEAM	12
5.0	SCATTERING OF A TEM_{01}^{*} BEAM	15
6.0	CONCLUSIONS	17
7.0	REFERENCES	18
	APPENDIX A	19
	APPENDIX B	22

1.0 INTRODUCTION

In infrared system design and optical countermeasures it is important to have available quantitative information on the extinction properties of aerosols, both of military and non-military origin, in the atmosphere. To obtain the extinction coefficient and the scattering phase function, one of the most basic physical quantities to be measured is the aerosol particle size. For this purpose light scattering from a laser beam is a technique commonly used. The standard theoretical tool in deducing the size of a spherical scatterer from the experimental result is the Mie theory developed at the turn of the present century. To ensure a sufficiently strong scattered signal, the incident electromagnetic field is often in the form of a collimated laser beam, whose characteristic width may be comparable with the size of the scatterer. In such cases the assumption in the Mie theory [1] that the incident field is an infinite plane wave is evidently not valid. The scattering of a Gaussian beam has been considered by Morita et al [2] but they have assumed that the scatterer was small compared to the beam waist. A more satisfactory approach to the problem has been given by Tsai and Pogorzelski [3]. Using the cylindrical vector solutions of the wave equation, they studied the scattering of a Gaussian beam from a laser operating in the fundamental mode by a spherical particle of arbitrary size. The particle, however, is assumed to be on the beam axis. In many physical applications - aerosol particle size measurement, for example - it is important to know the effects of the variation in intensity of the laser beam in a given sampling volume to properly design the experiment as well as to understand the measurements. Because of the lack of a more general theory, the interpretation of the results of a recent experiment [4] has to be limited to scatterers on the beam axis.

In this report we present theoretical results for the scattering of a Gaussian beam produced by a laser operating in the fundamental TEM_{00} mode. Some of the preliminary results have been presented in Ref. 5. The spherical scatterer is arbitrarily situated and its size is also arbitrary. For the measurement of aerosol particle size by intracavity scattering [6,7] the use of a beam in the TEM_{01}^{*} mode has been suggested. The power scattered and absorbed from such a beam by a spherical particle will also be given. This work was performed at DREV during 1976 under PCN 33A11 (formerly PCN 15B34), Aerosols Studies.

2.0 EXPANSION OF THE ELECTRIC FIELD OF A GAUSSIAN TEM_{00} BEAM

Before discussing the more general theory we briefly sketch the results of Ref. 3. It is assumed that the intensity of the electromagnetic beam $I(z=-z_0)$ measured at a plane $z=-z_0$ normal to the beam axis is given by

$$I(z=-z_0) \propto e^{-2r^2/w_0^2} \tag{1}$$

and the corresponding electric field is

$$\vec{E}(z=-z_0) = \vec{u}_x \, e^{-r^2/w_0^2} \, . \tag{2}$$

Since, as pointed out by Carter [8], (2) does not satisfy the Helmholtz equation except when r is small compared to w_0, it gives only the near axis behavior of $\vec{E}$ in the plane $z=-z_0$. In fact, Tsai and Pogorzelski [3] have shown that the electric field satisfying (2) near the beam axis can be derived from a Hertz potential:

$$\vec{\Pi} = \frac{j}{\omega} \, e^{-j\omega t} \sin\phi \, \frac{w_0^4}{4} \, \vec{u}_z \int_0^\infty e^{-\lambda^2 w_0^2/4} \, J_1(\lambda r) \, e^{jh(z+z_0)} \lambda^2 d\lambda \tag{3}$$

with $$\vec{E} = -\mu \frac{\partial}{\partial t} \vec{\nabla} \times \vec{\Pi} \tag{4}$$

In (3) the cylindrical coordinates (r, ϕ, z) refer to a frame with its origin on the beam axis, $h = k \sqrt{1-(\lambda/k)^2}$ and $k^2 = \mu\varepsilon\omega^2 + j\mu\sigma\omega$, where μ, ε and σ are respectively the permittivity, permeability and conductivity of the medium of propagation. Except that we have chosen a beam wave propagating in the positive z-direction, the symbols used here are the same as in Ref. 3. With (3) the magnetic field can also be written down from

$$\vec{H} = \vec{\nabla} \times \vec{\nabla} \times \vec{\Pi} \tag{5}$$

In fact, the electric field $\vec{E}$ given by (3) and (4) can be written as a linear combination of the first-order cylindrical wave eigenfunctions $\vec{m}_{o1\lambda}$ [9]

$$\vec{E} = \frac{w_0^4}{4} \int_0^\infty e^{-\lambda^2 w_0^2/4}\, \vec{m}_{o1\lambda}\, e^{jh(z+z_0)}\, \lambda^2 d\lambda \tag{6}$$

To solve the scattering problem with the spherical particle on the beam axis Tsai and Pogorzelski [3] established the following relationship between $\vec{m}_{o1\lambda}$ and the spherical wave eigenfunctions

$$e^{jhz}\, \vec{m}_{o1\lambda} = \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-1)!}{(\ell+1)!} j^{\ell-1} \lambda \left(j \frac{dP_\ell^{\,1}(\cos\alpha)}{d\alpha} \vec{m}_{o\ell 1} + \frac{P_\ell^{\,1}(\cos\alpha)}{\sin\alpha} \vec{n}_{e\ell 1} \right) \tag{7}$$

The scattered fields were then determined through the continuity conditions on the surface of the scatterer.

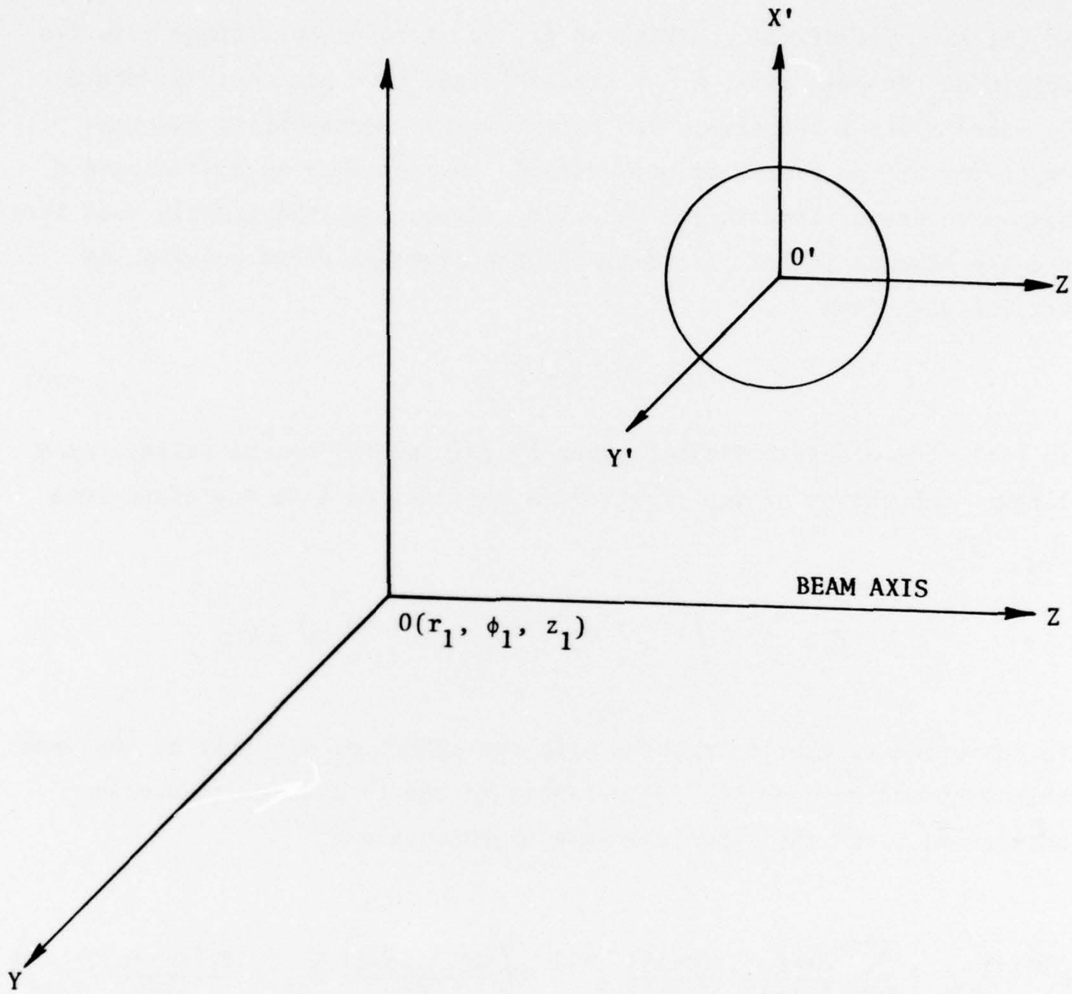


FIGURE 1 - Geometrical Configuration

For the general problem where the scatterer is not restricted to the beam axis, we choose the center of the spherical particle as the origin 0' of another coordinate system obtained from the old system by translation. Relative to the new system, the origin of the old system is situated at $0(r_1, \phi_1, z_1)$ (Fig. 1). To rewrite the Hertz potential of (3) in the new coordinates, the addition theorem for Bessel functions:

$$e^{jn\phi} J_n(\lambda r) = \sum_{m=-\infty}^{\infty} J_n(\lambda r_1)\, J_{n+m}(\lambda r')\, e^{j(m+n)\phi'}\, e^{-jm\phi_1} \tag{8}$$

can be used and

$$\vec{\Pi} = -\vec{u}_z \frac{w_0^4}{8\omega} e^{-j\omega t} \int_0^{\infty} e^{-\lambda^2 w_0^2/4}\, e^{jh(z'+z_1+z_0)} \times$$

$$\sum_{s=-\infty}^{\infty} J_s(\lambda r_1)\, e^{js(\phi'-\phi_1)} \left[J_{s+1}(\lambda r')\, e^{j\phi'} + J_{s-1}(\lambda r')\, e^{-j\phi'} \right] \lambda^2 d\lambda \tag{9}$$

It is more convenient for our purpose to combine the cylindrical vector eigenfunction of even and odd parities so that

$$\vec{m}_{s\lambda} = \vec{m}_{os\lambda} - j\, \vec{m}_{es\lambda},\quad \vec{n}_{s\lambda} = \vec{n}_{os\lambda} - j\, \vec{n}_{es\lambda} \tag{10}$$

The electric field $\vec{E}$ becomes

$$\vec{E} = \frac{jw_0^4}{8} e^{-j\omega t} \int_0^{\infty} e^{-\lambda^2 \omega_0^2/4}\, e^{jh(z'+z_1+z_0)} \times$$

$$\sum_{s=-\infty}^{\infty} J_s(\lambda r_1)\, e^{-js\phi_1} \left[\vec{m}_{s+1,\lambda} + \vec{m}_{s-1,\lambda} \right] \lambda^2 d\lambda \tag{11}$$

Unlike the electric field referring to an on-beam axis system, the field $\vec{E}$ now contains cylindrical vector waves of all orders.

To express (11) in terms of the spherical waves $\vec{m}_{\ell s}$, $\vec{n}_{\ell s}$ defined in a way similar to (10) it is necessary to generalize (7). It can be verified that for any integer s (Appendix A)

$$e^{jhz}\vec{m}_{s\lambda} = \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-s)!}{(\ell+s)!} j^{\ell-s+1} \lambda \left[\frac{dP_\ell^{s}(\cos\alpha)}{d\alpha} \vec{m}_{\ell s} + \frac{s}{\sin\alpha} P_\ell^{s} (\cos\alpha) \vec{n}_{\ell s} \right] \tag{12}$$

where P_ℓ^s are the Legendre polynomials.

Substituting (12) into (11) we obtain

$$\vec{E} = \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} D_A(\ell,s) \vec{m}_{\ell s} + D_B(\ell,s) \vec{n}_{\ell s} \tag{13}$$

where

$$\begin{Bmatrix} D_A(\ell,s) \\ D_B(\ell,s) \end{Bmatrix} = \frac{jw_0^{4}}{8} e^{-j\omega t} \int_0^{\infty} d\lambda e^{-\lambda^2 w_0^{2}/4} e^{jh(z_1+z_0)} \lambda^2 \times \left[J_{s-1}(\lambda r_1) e^{-j(s-1)\phi_1} + J_{s+1}(\lambda r_1) e^{-j(s+1)\phi_1} \right] \times \begin{Bmatrix} A(\ell,s;\lambda) \\ B(\ell,s;\lambda) \end{Bmatrix} \tag{14}$$

and

$$\begin{Bmatrix} A(\ell,s;\lambda) \\ B(\ell,s;\lambda) \end{Bmatrix} = \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-s)!}{(\ell+s)!} j^{\ell-s+1} \lambda \begin{Bmatrix} \dfrac{dP_\ell^s(\cos\alpha)}{d\alpha} \\ \dfrac{sP_\ell^s(\cos\alpha)}{\sin\alpha} \end{Bmatrix} \quad (15)$$

In an on-beam axis coordinate system, $\vec{E}$ is a linear combination of the spherical waves $\vec{m}_{\ell 1}$ and $\vec{n}_{\ell 1}$ only.

3.0 SCATTERING OF A GAUSSIAN TEM_{00} BEAM

To find the scattered fields $\vec{E}^r$, $\vec{H}^r$ produced by the spherical particle at 0', the continuity conditions on the surface of the scatterer (R = a) may be used

$$\vec{u}_R \times (\vec{E} + \vec{E}^r) = \vec{u}_R \times \vec{E}^t \quad (16)$$

$$\vec{u}_R \times (\vec{H} + \vec{H}^r) = \vec{u}_R \times \vec{H}^t \quad (17)$$

where $\vec{E}^t$, $\vec{H}^t$ are induced fields inside the sphere. For $\vec{E}^r$ and $\vec{E}^t$ we write

$$\vec{E}^r = \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} D_A(\ell,s)\, a^r_{\ell s}\, \vec{m}^{(3)}_{\ell s} + D_B(\ell,s)\, b^r_{\ell s}\, \vec{n}^{(3}_{\ell s} \quad (18)$$

and

$$\vec{E}^t = \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} D_A(\ell,s)\, a^t_{\ell s}\, \vec{m}^{(1)}_{\ell s} + D_B(\ell,s)\, b^t_{\ell s}\, \vec{n}^{(1)}_{\ell s} \quad (19)$$

The spherical waves $\vec{m}_{\ell s}^{(3)}$, $\vec{n}_{\ell s}^{(3)}$, $\vec{m}_{\ell s}^{(1)}$ and $\vec{n}_{\ell s}^{(1)}$ are obtained from the corresponding even and odd parity waves given in Ref. 7. The coefficients $a^r_{\ell s}$, $b^r_{\ell s}$, $a^t_{\ell s}$, $b^t_{\ell s}$ can now be determined from (16) and (17). It can be shown that the coefficients are independent of s such that

$$a^r_{\ell s} = a^r_{\ell s'}, \quad b^r_{\ell s} = b^r_{\ell s'} \tag{20}$$

$$a^t_{\ell s} = a^t_{\ell s'}, \quad b^t_{\ell s} = b^t_{\ell s'} \tag{21}$$

Furthermore, $a^r_{\ell s}$, $b^r_{\ell s}$ are just the well-known Mie coefficients given by (Appendix B)

$$a^r_{\ell s} = -\frac{j_\ell(\rho_1)\,[\rho j_\ell(\rho)]' - j_\ell(\rho)\,[\rho_1 j_\ell(\rho_1)]'}{j_\ell(\rho_1)\,[\rho h^{(1)}_\ell(\rho)]' - h^{(1)}_\ell(\rho)[\rho_1 j_\ell(\rho_1)]'} \tag{22}$$

$$b^r_{\ell s} = -\frac{j_\ell(\rho)[\rho_1 j_\ell(\rho_1)]' - (k'/k)^2 j_\ell(\rho_1)[\rho j_\ell(\rho)]'}{h^{(1)}_\ell(\rho)[\rho_1 j_\ell(\rho_1)]' - (k'/k)^2 j_\ell(\rho_1)[\rho h^{(1)}_\ell(\rho)]'} \tag{23}$$

where $\rho = ka$, $\rho_1 = k'a$ and k' is the propagation constant for the scatterer. Therefore, no scattering coefficient other than the Mie coefficients in (22) and (23) are involved in $\vec{E}^r$. Dropping the unnecessary subscripts s in $a^r_{\ell s}$ and $b^r_{\ell s}$ we can write

$$\vec{E}^r = \sum_{\ell=1}^{\infty}\left[a^r_\ell \sum_{s=-\infty}^{\infty} D_A(\ell,s)\,\vec{m}^{(3)}_{\ell s} + b^r_\ell \sum_{s=-\infty}^{\infty} D_B(\ell,s)\,\vec{n}^{(3)}_{\ell s}\right] \tag{24}$$

The magnetic fields $\vec{H}$ and $\vec{H}^r$ are given by

$$\vec{H} = \frac{k}{j\omega} \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} D_A(\ell,s)\, \vec{n}_{\ell s} + D_B(\ell,s)\, \vec{m}_{\ell s} \tag{25}$$

and

$$\vec{H}^r = \frac{k}{j\omega} \sum_{\ell=1}^{\infty} \left[a_\ell^r \sum_{s=-\infty}^{\infty} D_A(\ell,s)\, \vec{n}_{\ell s}^{(3)} + b_\ell^r \sum_{s=-\infty}^{\infty} D_B(\ell,s)\, \vec{m}_{\ell s}^{(3)} \right] \tag{26}$$

With the expressions (24) and (26) for the scattered fields we can write down at once a generalized expression for the power scattered in a given direction.

The rate W_s at which energy is being scattered by the particle can now be obtained by integrating the Poynting vector corresponding to $\vec{E}^r$ and $\vec{H}^r$ over the surface of a large sphere of radius R centered at 0'. In fact,

$$W_s = \lim_{R\to\infty} \tfrac{1}{2} \int_0^{\pi} \int_0^{2\pi} R^2 \left(E_\theta^r H_\phi^{r*} - E_\phi^r H_\theta^{r*}\right) \sin\theta\, d\theta\, d\phi$$

$$= \frac{2\pi}{k\omega} \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} \frac{\ell(\ell+1)}{2\ell+1} \frac{(\ell+s)!}{(\ell-s)!} \left\{ |a_\ell^r D_A(\ell,s)|^2 + |b_\ell^r D_B(\ell,s)|^2 \right\} \tag{27}$$

In deriving (27), we used the following relations

$$\int_0^{\pi}\int_0^{2\pi}\left(m^{(3)}_{\ell s\theta}\, m^{(3)*}_{\ell' s'\phi} - m^{(3)}_{\ell s\phi}\, m^{(3)*}_{\ell' s'\theta}\right)\sin\theta d\theta d\phi = 0 \tag{28}$$

$$\int_0^{\pi}\int_0^{2\pi}\left(n^{(3)}_{\ell s\phi}\, n^{(3)*}_{\ell' s'\theta} - n^{(3)}_{\ell s\theta}\, n^{(3)*}_{\ell' s'\phi}\right)\sin\theta d\theta d\phi = 0 \tag{29}$$

and

$$\int_0^{\pi}\int_0^{2\pi}\left(m^{(3)}_{\ell s\theta}\, n^{(3)*}_{\ell' s'\phi} - m^{(3)}_{\ell s\phi}\, n^{(3)*}_{\ell' s'\theta}\right)\sin\theta d\theta d\phi$$

$$= \delta_{\ell,\ell'}\delta_{s,s'}\, \frac{4\pi\ell}{2\ell+1}\,(\ell+1)\,\frac{(\ell+s)!}{(\ell-s)!}\, h^{(1)}_{\ell}(\rho)\,\frac{1}{\rho}\,\frac{d}{d\rho}\,[\rho h^{(1)*}_{\ell}(\rho)] \tag{30}$$

In a similar manner, the rate W_t at which energy is being scattered and absorbed by the particle can be found by evaluating the integral

$$W_t = \lim_{R\to\infty} -\tfrac{1}{2}\,\mathrm{Re}\int_0^{\pi}\int_0^{2\pi} R^2\sin\theta\,(E_\theta H^{r*}_{\phi} + E^{r}_{\theta}H^{*}_{\phi} - E_\phi H^{r*}_{\theta} - E^{r}_{\phi}H^{*}_{\theta})\,d\theta d\phi \tag{31}$$

to give

$$W_t = \frac{2\pi}{k\omega}\,\mathrm{Re}\sum_{\ell=1}^{\infty}\sum_{s=-\infty}^{\infty}\frac{\ell(\ell+1)}{2\ell+1}\,\frac{(\ell+s)!}{(\ell-s)!}\left\{a^{r}_{\ell}|D_A(\ell,s)|^2 + b^{r}_{\ell}|D_B(\ell,s)|^2\right\} \tag{32}$$

For the special case when the beam axis passes through the center of the spherical scatterer $r_1 = 0$, (14) becomes

$$\begin{Bmatrix} D_A(\ell,s) \\ D_B(\ell,s) \end{Bmatrix} = \frac{jw_0^4}{8} e^{-j\omega t} \int_0^\infty e^{-\lambda^2 w_0^2/4} e^{jh(z_1+z_0)} \times$$

$$(\delta_{s,1} + \delta_{s,-1}) \times \begin{Bmatrix} A(\ell,s;\lambda) \\ B(\ell,s;\lambda) \end{Bmatrix} \lambda^2 d\lambda \tag{33}$$

After some straightforward algebraic manipulation we arrive at the following results for on-beam axis scattering

$$W_s = \frac{2\pi}{k\omega} \sum_{\ell=1}^{\infty} \frac{2(2\ell+1)}{[\ell(\ell+1)]^2} \left[|d_A(\ell) a_\ell^r|^2 + |d_B(\ell)\, b_\ell^r|^2 \right] \tag{34}$$

and

$$W_t = \frac{2\pi}{k\omega} \sum_{\ell=1}^{\infty} \frac{2(2\ell+1)}{[\ell(\ell+1)]^2} \left[a_\ell^r |d_A(\ell)|^2 + b_\ell^r |d_B(\ell)|^2 \right] \tag{35}$$

where

$$d_A(\ell) = \frac{w_0^4}{8} \int_0^\infty e^{-\lambda^2 w_0^2/4} e^{jh(z_1+z_0)} \frac{dP^1_\ell(\cos\alpha)}{d\alpha} \lambda^2 d\lambda \tag{36}$$

and

$$d_B(\ell) = \frac{w_0^4}{8} \int_0^\infty e^{-\lambda^2 w_0^2/4} e^{jh(z_1+z_0)} \frac{P^1_\ell(\cos\alpha)}{\sin\alpha} \lambda^2 d\lambda \tag{37}$$

In the infinite plane wave limit corresponding to $w_0 \to \infty$, both $d_A(\ell)$ and $d_B(\ell)$ approach to a common value $\ell(\ell+1)/2$ and the well-known results

$$W_s = \frac{\pi}{k\omega} \sum_{\ell=1}^{\infty} (2\ell+1) \left(|a_\ell^r|^2 + |b_\ell^r|^2 \right) \tag{38}$$

and

$$W_t = \frac{\pi}{k\omega} \operatorname{Re} \sum_{\ell=1}^{\infty} (2\ell+1) \left(a_\ell^r + b_\ell^r \right) \tag{39}$$

are obtained.

4.0 EXPANSION OF THE ELECTRIC FIELD OF A TEM_{01}^* BEAM

The first-order oscillation mode in a laser cavity with cylindrical symmetry about the beam axis is often referred to as the TEM_{01}^* mode. Its intensity distribution in a plane normal to the beam axis is

$$I(z=-z_0) \propto r^2 e^{-2r^2/w_0^2} \tag{40}$$

and the electric field near the beam axis is

$$\vec{E}(z=-z_0) = \vec{u}_x \, r \, e^{-r^2/w_0^2} \tag{41}$$

In contradistinction to the fundamental mode discussed above, such a field cannot be derived from a single Hertz potential. Two complementary Hertz potentials $\vec{\Pi}^{(1)}$ and $\vec{\Pi}^{(2)}$ of the electric and magnetic types [9] are required and the corresponding partial fields are given by

$$\vec{E}^{(1)} = \vec{\nabla} \times \vec{\nabla} \times \vec{\Pi}^{(1)}, \qquad \vec{H}^{(1)} = \varepsilon \frac{\partial}{\partial t} \vec{\nabla} \times \vec{\Pi}^{(1)} \tag{42}$$

and

$$\vec{E}^{(2)} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{\Pi}^{(2)}, \quad \vec{H}^{(2)} = \vec{\nabla} \times \vec{\nabla} \times \vec{\Pi}^{(2)} \tag{43}$$

From the general theory of vector potentials, it is sufficient to consider $\vec{\Pi}^{(1)}$, $\vec{\Pi}^{(2)}$ of the form

$$\vec{\Pi}^{(1)} = \psi \vec{u}_z, \quad \vec{\Pi}^{(2)} = \chi \vec{u}_z \tag{44}$$

An immediate consequence of using a Hertz potential of the electric type is that $\vec{E}$ is not completely transverse. However, the longitudinal component of $\vec{E}$ does not contribute to $I(z=-z_0)$ and it will be seen that the field $\vec{E}$ is dominated by its transverse component near the beam axis. The Hertz potentials ψ, χ can be expressed as

$$\psi = e^{-j\omega t} \frac{r^2}{2} \cos^2\phi \; e^{-r^2/w_0^2} f(z) \tag{45}$$

$$\chi = \frac{1}{2\omega} e^{-j\omega t} r^2 \sin^2\phi \; e^{-r^2/w_0^2} g(z) \tag{46}$$

The functions $f(z)$, $g(z)$ are chosen such that ψ and χ satisfy the scalar wave equation as well as the boundary conditions $\left.\frac{df}{dz}\right|_{z=-z_0} = 1$, $g(z=-z_0) = 1$. As in the case of the fundamental mode, ψ and χ can be expressed in terms of the solutions of the scalar wave equation in cylindrical coordinates. Let us define the functions $q_2(\lambda)$, $q_0(\lambda)$ and $q_{-2}(\lambda)$ to be

$$q_{s_i}(\lambda) = \int_0^\infty r^2 \, e^{-r^2/w_0^2} \, J_{s_i}(\lambda) \, r dr \tag{47}$$

where i=1, 2, 3 and $s_1=2$, $s_2=0$, $s_3=-2$. Using the boundary conditions of ψ and χ and the Fourier-Bessel theorem, we can obtain

$$\psi = \sum_{i=1}^{3} e^{-j\omega t+js_i\phi} \frac{\delta_i}{8} \int_0^\infty \frac{q_{s_i}(\lambda)}{h} J_{s_i}(\lambda r)\, e^{jh(z+z_0)}\, \lambda d\lambda \tag{48}$$

and

$$\chi = -\sum_{i=1}^{3} e^{-j\omega t+js_i\phi} \frac{(-1)^i\delta_i}{8\omega} \int_0^\infty q_{s_i}(\lambda)\, J_{s_i}(\lambda r)\, e^{jh(z+z_0)}\lambda d\lambda \tag{49}$$

where $\delta_1=\delta_3=1$, $\delta_2=2$.

The partial fields $\vec{E}^{(1)}$ and $\vec{E}^{(2)}$ are then given by

$$\vec{E}^{(1)} = e^{-j\omega t} \sum_{i=1}^{3} e^{js_i\phi} \frac{\delta_i}{8} \int_0^\infty q_{s_i}(\lambda)\, e^{jh(z+z_0)} \times$$

$$\left\{ j \frac{\partial J_{s_i}(\lambda r)}{\partial r} \vec{u}_r - \frac{s_i}{r} J_{s_i}(\lambda r)\, \vec{u}_\phi + \frac{\lambda^2}{h} J_{s_i}(\lambda r)\, \vec{u}_z \right\} \lambda d\lambda \tag{50}$$

$$\vec{E}^{(2)} = -e^{-j\omega t} \sum_{i=1}^{3} e^{js_i\phi} \frac{(-1)^i\delta_i j}{8} \int_0^\infty q_{s_i}(\lambda)\, e^{jh(z+z_0)} \times$$

$$\left\{ -\frac{js_i}{r} J_{s_i}(\lambda r)\, \vec{u}_r + \frac{\partial J_{s_i}(\lambda r)}{\partial r} \vec{u}_\phi \right\} \lambda d\lambda \tag{51}$$

The partial field $\vec{E}^{(2)}$ is transverse but $\vec{E}^{(1)}$ has a longitudinal component. It is clear from (50) that near the beam axis ($r \to 0$) the transverse component dominates. Both $\vec{E}^{(1)}$ and $\vec{E}^{(2)}$ can now be expressed in terms of the cylindrical eigenfunctions so that

$$\vec{E} = \vec{E}^{(1)} + \vec{E}^{(2)} = e^{-j\omega t} \sum_{i=1}^{3} \frac{\delta_i j}{8} \int_0^\infty q_{s_i}(\lambda)\, e^{jh(z+z_0)} \times$$

$$\left\{ \frac{k}{h} \vec{n}_{s_i\lambda} + (-1)^i\, j\vec{m}_{s_i\lambda} \right\} \lambda d\lambda \,. \qquad (52)$$

5.0 SCATTERING OF A TEM_{01}^{*} BEAM

The electric field $\vec{E}$ in (52) is expressed in terms of the on-beam axis coordinates. To find the fields scattered by an arbitrarily situated sphere we again introduce an off-axis system with coordinates (r', ϕ', z') as before. By applying the addition theorem for Bessel functions, the field $\vec{E}$ in the off-axis system becomes

$$\vec{E} = \frac{e^{-j\omega t}}{8} \sum_{i=1}^{3} \sum_{s=-\infty}^{\infty} \int_0^\infty \xi(s,i,\lambda)\, e^{jhz'} \left[\frac{jk}{h} \vec{n}_{s+s_i,\lambda} - (-1)^i \vec{m}_{s+s_i,\lambda} \right] \lambda d\lambda \qquad (53)$$

where

$$\xi(s,i,\lambda) = \delta_i \lambda q_{s_i}(\lambda)\, e^{jh(z_1+z_0)}\, J_s(\lambda r_1)\, e^{-js\phi_1} \qquad (54)$$

and the coordinate variables in $\vec{m}_{s\lambda}$, $\vec{n}_{s\lambda}$ in (53) are (r', ϕ', z').

To transform the cylindrical wave functions $\vec{m}_{s\lambda}$, $\vec{n}_{s\lambda}$ to the spherical functions, we need an identity for $\vec{n}_{s\lambda}$ similar to that for $\vec{m}_{s\lambda}$ given in (12). It can be shown that (Appendix A)

$$e^{jhz}\,\vec{n}_{s\lambda} = \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-s)!}{(\ell+s)!}\, j^{\ell-s+1}\lambda \left[\frac{dP^{s}{}_{\ell}(\cos\alpha)}{d\alpha}\,\vec{m}_{\ell s} + \frac{s}{\sin\alpha}\,P^{s}{}_{\ell}(\cos\alpha)\,\vec{n}_{\ell s}\right] \tag{55}$$

and hence

$$\vec{E} = \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} \left[J(\ell,s)\,\vec{m}_{\ell s} + K(\ell,s)\,\vec{n}_{\ell s}\right] \tag{56}$$

where

$$J(\ell,s) = \frac{e^{-j\omega t}}{8} \sum_{i=1}^{3} \int_{0}^{\infty} \xi(s-s_i,i,\lambda) \left[\frac{jk}{h} B(\ell,s;\lambda) - (-1)^{i} A(\ell,s;\lambda)\right] d\lambda \tag{57}$$

and $K(\ell,s)$ is obtained from $J(\ell,s)$ by interchanging $A(\ell,s;\lambda)$ with $B(\ell,s;\lambda)$. Because $\vec{E}$ in (56) has exactly the same form as (13), the scattered fields can be written down at once

$$\vec{E}^{r} = \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} \left[a^{r}{}_{\ell}\, J(\ell,s)\,\vec{m}^{(3)}{}_{\ell s} + b^{r}{}_{\ell}\, K(\ell,s)\,\vec{n}^{(3)}{}_{\ell s}\right] \tag{58}$$

and

$$\vec{H}^{r} = \frac{k}{j\omega} \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} \left[a^{r}{}_{\ell}\, J(\ell,s)\,\vec{n}^{(3)}{}_{\ell s} + b^{r}{}_{\ell}\, K(\ell,s)\,\vec{n}^{(3)}{}_{\ell s}\right] \tag{59}$$

The rates at which energy is being scattered away from the beam are

$$W_s = \frac{2\pi}{k\omega} \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} \frac{\ell(\ell+1)}{2\ell+1} \frac{(\ell+s)!}{(\ell-s)!} \left\{ |a^r_\ell \, J(\ell,s)|^2 + |b^r_\ell \, K(\ell,s)|^2 \right\} \tag{60}$$

and W_t, the rate at which energy is scattered and absorbed, is

$$W_t = \frac{2\pi}{k\omega} \, \mathrm{Re} \sum_{\ell=1}^{\infty} \sum_{s=-\infty}^{\infty} \frac{\ell(\ell+1)}{2\ell+1} \frac{(\ell+s)!}{(\ell-s)!} \left\{ a^r_\ell |J(\ell,s)|^2 + b^r_\ell |K(\ell,s)|^2 \right\} \tag{61}$$

The coefficients a^r_ℓ, b^r_ℓ in (58) - (61) are again the Mie coefficients given in (22) and (23).

6.0 CONCLUSIONS

The results of Tsai and Pogorzelski [3] derived for a Gaussian beam corresponding to the fundamental mode TEM_{00} have been generalized. In the first place, the incident fields are expressed in terms of the cylindrical and spherical vector wave eigenfunctions respectively in an off-beam-axis coordinate system. This enables us to obtain the scattered fields as well as the powers scattered and absorbed from the beam by an arbitrarily situated spherical object. We have shown that although the incident and the scattered fields in an off-beam-axis system are linear combinations of spherical wave functions of all orders (i.e. $\vec{m}_{\ell s}$, $\vec{n}_{\ell s}$ with s equal to any integer), the scattering coefficients are independent of s and are just the Mie coefficients.

Parallel results are also obtained for the cylindrical symmetric mode TEM^*_{01}. These theoretical results together with a standard program for calculating Mie coefficients [10] can be readily applied to numerical computations for the scattering of beam waves without imposing restrictions on the size or position of the scatterer. With the more general theory, aerosol particle sizing can be more efficiently carried out using the laser scattering technique.

7.0 REFERENCES

[1] Mie, G., "Beiträge zur Optik trüber Medien, speziell kolloidaler Metallösungen", Ann. Phys., Vol. 25, pp. 377-445, 1908.

[2] Morita, N., Tanaka, T., Yamasaki, T. and Nakanishi, Y., "Scattering of a beam wave by a spherical object", IEEE AP, Vol. 16, pp. 724-727, 1968.

[3] Tsai, W.C. and Pogorzelski, R.J., "Eigenfunction solution of the scattering of beam radiation field by spherical objects", J. Opt. Soc. Am., Vol. 65, p. 1457, 1975.

[4] Casperson, L.W., Yeh, C. and Yeung, W.F., "Single particle scattering with focused laser beams", App. Opt., Vol. 16, pp. 1104-1107, 1977.

[5] Tam, W.G., "Off beam axis scattering by spherical particles", App. Opt., Vol. 16, pp. 2016-2018, 1977. This article presents some of the results of the present report.

[6] Schuster, B.G. and Knollenberg, R., "Detection and sizing of small particles in an open cavity gas laser", App. Opt. Vol. 11, pp. 1515-1520, 1972.

[7] Tam, W.G. and Corriveau, R., "On the method of laser intracavity scattering for aerosol size measurement", Proceedings of the Symposium on Radiation in the Atmosphere, Garmisch - Partenkirchen, FRG, 19-28 August 1976.

[8] Carter, W.H., "Electromagnetic field of a Gaussian beam with an elliptical cross section", J. Opt. Soc. Am., Vol. 62, pp. 1195-1201, 1972.

[9] Stratton, J.A., "Electromagnetic Theory", McGraw Hill, New York, 1941.

[10] Dave, J.V., "Subroutine for computing the parameters of the electromagnetic radiation scattered by a sphere", Rep. No. 320-3237, IBM Scientific Center, Palo Alto, California, 1968.

APPENDIX A

In this appendix we sketch the proofs of the two formulas:

$$e^{jhz}\,\vec{m}_{s\lambda} = \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)}\,\frac{(\ell-s)!}{(\ell+s)!}\,j^{\ell-s+1}\lambda\left[\frac{dP_{\ell}^{s}(\cos\alpha)}{d\alpha}\,\vec{m}_{\ell s} + \frac{s}{\sin\alpha}\,P_{\ell}^{s}(\cos\alpha)\,\vec{n}_{\ell s}\right] \tag{A-1}$$

and

$$e^{jhz}\,\vec{n}_{s\lambda} = \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)}\,\frac{(\ell-s)!}{(\ell+s)!}\,j^{\ell-s+1}\lambda\left[\frac{dP_{\ell}^{s}(\cos\alpha)}{d\alpha}\,\vec{n}_{\ell s} + \frac{s}{\sin\alpha}\,P_{\ell}^{s}(\cos\alpha)\,\vec{m}_{\ell s}\right] \tag{A-2}$$

We begin by proving (A-2) from which (A-1) can readily be obtained. Let us denote the vector $e^{-js\phi}\,e^{jhz}\,\vec{n}_{s\lambda}$ by $\vec{N}$ and its component along $\vec{u}_R$ by N_R. Then from eq. (2.14) we have

$$N_R = \frac{e^{jhz}}{k}\left(h\,\frac{\partial J_s(\lambda r)}{\partial r}\,\sin\theta - j\lambda^2\,J_s(\lambda r)\,\cos\theta\right) \tag{A-3}$$

Using the identity

$$J_s(\lambda r)\,e^{jhz} = \sum_{\ell=0}^{\infty} j^{\ell-s}(2\ell+1)\,\frac{(\ell-s)!}{(\ell+s)!}\,P_{\ell}^{s}(\cos\alpha)\,P_{\ell}^{s}(\cos\theta)\,j_{\ell}(kR) \tag{A-4}$$

and recurrence relations of $J_s(\lambda r)$, $j_{\ell}(kR)$, $p_{\ell}^{s}(\cos\alpha)$ and $P_{\ell}^{s}(\cos\theta)$, we can show after rather lengthy but elementary manipulations that

$$N_R = \sum_{\ell=1}^{\infty} j^{\ell-s+1} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-s)!}{(\ell+s)!} \lambda \frac{dP_\ell^s(\cos\alpha)}{d\alpha} n_{\ell sR}. \qquad \text{(A-5)}$$

Hence $\vec{N}'$ defined by

$$\vec{N}' = \vec{N} - \sum_{\ell=1}^{\infty} j^{\ell-s+1} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-s)!}{(\ell+s)!} \lambda \frac{dP_\ell^s(\cos\alpha)}{d\alpha} \vec{n}_{\ell s} \qquad \text{(A-6)}$$

is a vector which has no component along $\vec{u}_R$. In a similar manner, the $\vec{u}_\phi$ component of $\vec{N}'$ can be proved to be given by

$$N_\phi' = \sum_{\ell=1}^{\infty} j^{\ell-s+1} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-s)!}{(\ell+s)!} \frac{\lambda s}{\sin\alpha} P_\ell^s (\cos\alpha) m_{\ell s\phi}. \qquad \text{(A-7)}$$

From the fact that $\vec{m}_{\ell s}$ contains no $\vec{u}_R$ component, it follows that

$$\vec{N}'' \equiv \vec{N} - \sum_{\ell=1}^{\infty} j^{\ell-s+1} \frac{2\ell+1}{\ell(\ell+1)} \frac{(\ell-s)!}{(\ell+s)!} \lambda \left[\frac{dP_\ell^s(\cos\alpha)}{d\alpha} \vec{n}_{\ell s} + \frac{s}{\sin\alpha} P_\ell^s(\cos\alpha) \vec{m}_{\ell s}\right] \qquad \text{(A-8)}$$

may only have a non-zero component along $\vec{u}_\theta$. To complete the proof that $\vec{N}''$ is indeed a null vector, we note that since $\vec{N}''$ must be divergence free (as $\vec{N}$, $\vec{n}_{\ell s}$, $\vec{m}_{\ell s}$ are all divergence free) it can only be expressed as a linear combination of $\vec{m}_{\ell s}$ and $\vec{n}_{\ell s}$ i.e.

$$\vec{N}'' = \sum_{\ell=0}^{\infty} \alpha_\ell \vec{n}_{\ell s} + \beta_\ell \vec{n}_{\ell s}. \tag{A-9}$$

In (A-9) only $\vec{n}_{\ell s}$ $(0 \leq \ell \leq \infty)$ has $\vec{u}_R$ component and hence $\beta_\ell = 0$ following from the linear independence of the vectors $\vec{n}_{\ell s}$. Similarly, since $\vec{m}_{\ell s}$ has $\vec{u}_q$ components while $\vec{N}''$ does not, the coefficients α_ℓ must also vanish. This proves (A-2) and (A-1) can be obtained at once from (A-2) and the fact

$$\nabla \times (e^{jhz} \vec{n}_{s\lambda}) = ke^{jhz} \vec{m}_{s\lambda} \tag{A-10}$$

$$\nabla \times \vec{m}_{\ell s} = k \vec{n}_{\ell s} \quad \text{and}$$

$$\nabla \times \vec{n}_{\ell s} = k \vec{m}_{\ell s}.$$

APPENDIX B

Let

$$\vec{u}_R \times \vec{n}_{\ell s} = \vec{M}_{\ell s}\ , \quad \vec{u}_R \times \vec{n}_{\ell s} = \vec{N}_{\ell s}$$

From the boundary conditions (16, 17) on the spherical surface R = a, we obtain the following equations:

$$\sum_{\ell} \sum_{s} D_A(\ell,s) \left(\vec{M}_{\ell s} + a^r_{\ell s} \vec{M}^{(3)}_{\ell s} - a^t_{\ell s} \vec{M}^{(1)}_{\ell s} \right) \Big|_{R=a}$$

$$+ D_B(\ell,s) \left(\vec{N}_{\ell s} + b^r_{\ell s} \vec{N}^{(3)}_{\ell s} - b^t_{\ell s} \vec{N}^{(1)}_{\ell s} \right) \Big|_{R=a} = 0 \qquad \text{(B-1)}$$

and

$$\sum_{\ell} \sum_{s} D_A(\ell,s) \left(\vec{N}_{\ell s} + a^r_{\ell s} \vec{N}^{(3)}_{\ell s} - \frac{k'}{k} a^t_{\ell s} \vec{N}^{(1)}_{\ell s} \right) \Big|_{R=a}$$

$$+ D_B(\ell,s) \left(\vec{M}_{\ell s} + b^r_{\ell s} \vec{M}^{(3)}_{\ell s} - \frac{k'}{k} b^t_{\ell s} \vec{M}^{(1)}_{\ell s} \right) \Big|_{R=a} = 0 \qquad \text{(B-2)}$$

If we write

$$\vec{m}_{\ell s} = m_{\ell s\theta} \vec{u}_\theta + m_{\ell s\phi} \vec{u}_\phi \qquad \text{(B-3)}$$

and $$\vec{n}_{\ell s} = n_{\ell sR} \vec{u}_R + n_{\ell s\theta} \vec{u}_\theta + n_{\ell s\phi} \vec{u}_\phi \qquad \text{(B-4)}$$

then $$\vec{M}_{\ell s} = - m_{\ell s\phi} \vec{u}_\theta + m_{\ell s\theta} \vec{u}_\phi \qquad \text{(B-5)}$$

$$\vec{N}_{\ell s} = - n_{\ell s \phi}\vec{u}_\theta + n_{\ell s \theta}\vec{u}_\phi \tag{B-6}$$

It can be seen at once that

$$\vec{m}_{\ell s} \cdot \vec{n}_{\ell ' s '} = \vec{M}_{\ell s} \cdot \vec{N}_{\ell ' s '} \tag{B-7}$$

and
$$\vec{m}_{\ell s} \cdot \vec{m}_{\ell ' s '} = \vec{M}_{\ell s} \cdot \vec{M}_{\ell ' s '} \tag{B-8}$$

Furthermore, using the orthogonal properties of $\vec{m}_{\ell s}$ and $\vec{n}_{\ell s}$, one can prove that

$$\int_0^{2\pi} \int_0^{\pi} \vec{M}_{\ell s} \cdot \vec{N}^*_{\ell ' s '} \sin\theta d\phi d\theta = 0 \tag{B-9}$$

$$\int_0^{2\pi} \int_0^{\pi} \vec{M}_{\ell s} \cdot \vec{M}^*_{\ell ' s '} \sin\theta d\phi d\theta$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \left(\vec{m}_{o\ell s} \cdot \vec{m}^*_{o\ell ' s '} + \vec{m}_{e\ell s} \cdot \vec{m}^*_{e\ell ' s '} \right)$$

$$= \delta_{\ell,\ell '}\delta_{s,s '} (1+\delta) \frac{4\pi}{2\ell+1} \frac{(\ell+s)!}{(\ell-s)!} \ell(\ell+1) \left| z_\ell(kR) \right|^2 \tag{B-10}$$

and

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \vec{N}_{\ell s} \cdot \vec{N}^*_{\ell ' s '}$$

$$= \delta_{\ell,\ell '}\delta_{s,s '} (1+\delta) \frac{4\pi}{2\ell+1} \frac{(\ell+s)!}{(\ell-s)!} \ell(\ell+1) \left| \frac{1}{kR} \frac{\partial}{\partial R} [Rz_\ell(kR)] \right|^2 \tag{B-11}$$

In eqs (B-10) and (B-11)

$$\delta = \begin{cases} 0 & \text{for } s \neq 0 \\ 1 & \phantom{\text{for }} s = 0 \end{cases}$$

since the two sets of vectors $\left\{\vec{M}_{\ell s}, \vec{M}^{(1)}_{\ell s}, \vec{M}^{(3)}_{\ell s}\right\}$ and $\left\{\vec{N}_{\ell s}, \vec{N}^{(1)}_{\ell s}, \vec{N}^{(3)}_{\ell s}\right\}$ have respectively the same angular dependence. Relations similar to eqs (B-9) - (B-11) can readily be written down for any two of the six vectors. From these relations and eqs (B-1) and (B-2), we obtain that

$$j_\ell(ka) = a^r_{\ell s} h^{(1)}_\ell(ka) - a^t_{\ell s} j_\ell(k'a) = 0 \tag{B-12}$$

$$[\rho j_\ell(\rho)]' + a^r_{\ell s}[\rho h^{(1)}_\ell(\rho)]' - a^t_{\ell s}[\rho_1 j_\ell(\rho_1)]' = 0 \tag{B-13}$$

$$j_\ell(\rho) + b^r_{\ell s} h_\ell(\rho) - \frac{k'}{k} b^t_{\ell s} j_\ell(\rho_1) = 0 \tag{B-14}$$

$$[\rho j_\ell(\rho)]' + b^r_{\ell s} [\rho h^{(1)}_\ell(\rho)]' - \frac{k}{k'} b^*_{\ell s} [\rho_1 j_\ell(\rho_1)]' = 0 \tag{B-15}$$

where $\rho = ka$, $\rho_1 = k'a$.

It is clear that the unknowns $a^r_{\ell s}$, $b^r_{\ell s}$, $a^t_{\ell s}$ and $b^t_{\ell s}$ in the above equations are independent of the values of s. Furthermore, for any arbitrary integers

$$a^r_{\ell s} = - \frac{j_\ell(\rho_1)\,[\rho j_\ell(\rho)]' - j_\ell(\rho)\,[\rho_1 j_\ell(\rho_1)]'}{j_\ell(\rho_1)\,[\rho h^{(1)}_\ell(\rho)]' - h^{(1)}_\ell(\rho)\,[\rho_1 j_\ell(\rho_1)]'} \tag{B-16}$$

$$b_{\ell s}^{r} = - \frac{j_\ell(\rho)\ [\rho_1 j_\ell(\rho_1)]' - (k'/k)^2\ j_\ell(\rho_1)\ [\rho j_\ell(\rho)]'}{h_\ell^{(1)}(\rho)\ [\rho_1 j_\ell(\rho_1)]' - (k'/k)^2\ j_\ell(\rho_1)\ [\rho h_\ell^{(1)}(\rho)\ ]'} \qquad \text{(B-17)}$$

which are just the Mie scattering coefficients.

78